4 Deriving Equations of Motion

Although the possible mechanisms for storing energy in latex bands is nearly infinite, three common types of band configuration have emerged:

4.1 Conventional Spearguns

Are the simplest and most common form of spearguns. They use rubber slings to directly power the shaft; there are no pulleys or rollers. No pretension is possible; this means the bands are slack for 25-30% of the speargun's length per Figure X.

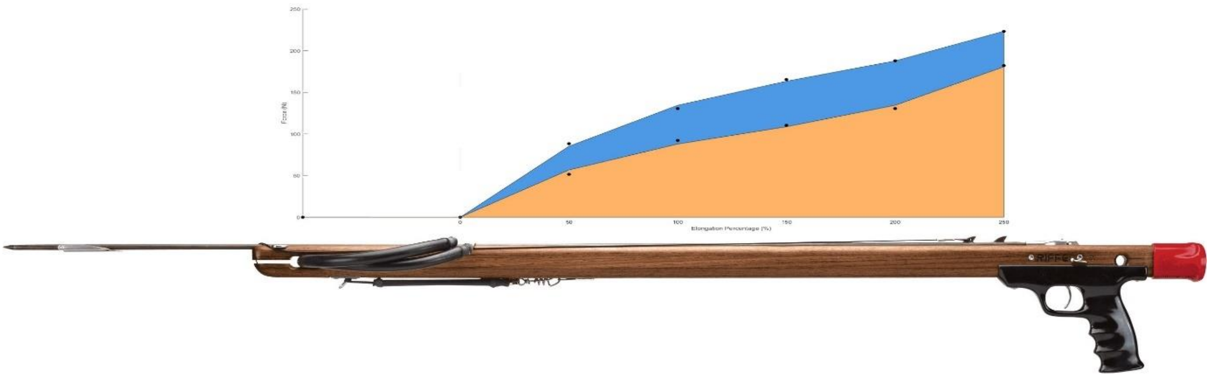


Figure 23 - Conventional Speargun Power Diagram

Since the bands are fixed at the muzzle and the spear end, the velocity of the centre of gravity of the bands is equal to half the velocity of the spear itself.

$$V_{bands} = 0.5\, V_{spear} \tag{4-1}$$

The bands and spear move forward (positive) and the gun recoils backwards (negative). Equating the horizontal momentum can give an expression for spear velocity in terms of gun velocity:

$$\Delta\, P_{system} = 0$$

$$M_{gun}V_{gun} = (M_{bands}V_{bands}) + \ (M_{spear}\, V_{spear})$$

Substituting equation 5-1 into the momentum balance below:

$$M_{gun}V_{gun} = \left(M_{bands} \times 0.5\, V_{spear}\right) + \ (M_{spear}V_{spear})$$

$$M_{gun}V_{gun} = \left(M_{bands} \times 0.5\, V_{spear}\right) + \ (M_{spear}V_{spear})$$

$$M_{gun}\,V_{gun} = V_{spear}\left[\frac{M_{bands}}{2} + M_{spear}\right]$$

$$V_{spear} = \frac{M_{gun}\,V_{gun}}{\left[\frac{M_{bands}}{2} + M_{spear}\right]} \qquad \textbf{(4-2)}$$

The kinetic energy in the system is also conserved in the horizontal plane:

$$\Delta\,E_{system} = 0$$

$$E_{system} = \frac{M_{gun}V_{gun}^{\;2}}{2} + \frac{M_{bands}V_{bands}^{\;2}}{2} + \frac{M_{spear}V_{spear}^{\;2}}{2}$$

Substituting equation5-1 into the energy balance below:

$$2E_{system} = M_{gun}V_{gun}^{\;2} + M_{bands}(0.5\,V_{spear})^2 + M_{spear}V_{spear}^{\;2}$$

$$M_{gun}V_{gun}^{\;2} + 0.25\,M_{bands}V_{spear}^{\;2} + M_{spear}V_{spear}^{\;2} - 2E_{system} = 0$$

$$M_{gun}V_{gun}^{\;2} + V_{spear}^{\;2}\,(\,0.25\,M_{bands} + M_{spear}) - 2E_{system} = 0$$

Substituting equation 5-2’s expression for spear velocity results an expression for recoil velocity in terms of component masses and system energy:

$$M_{gun}V_{gun}^{\;2} + \left[\frac{M_{gun}\,V_{gun}}{\left(\frac{M_{bands}}{2} + M_{spear}\right)}\right]^2 \left[\frac{M_{bands}}{4} + M_{spear}\right] - 2E_{system} = 0 \qquad \textbf{(4-3)}$$

Equation 5-3 can be solved for gun velocity. The result is fed into equation 5-2 to find spear velocity and the result halved to find band velocity per equation 5-1.

All results can be checked by adding component kinetic energies and comparing the result to system energy.

4.2 Roller Spearguns

Roller spearguns have a roller at the muzzle; the distance available for band stretch is significantly more than conventional spearguns. ‘*Preload*’ or a pre-stretched band is possible for roller spearguns which means the lower limit of percentage elongation is greater than zero per Figure X:

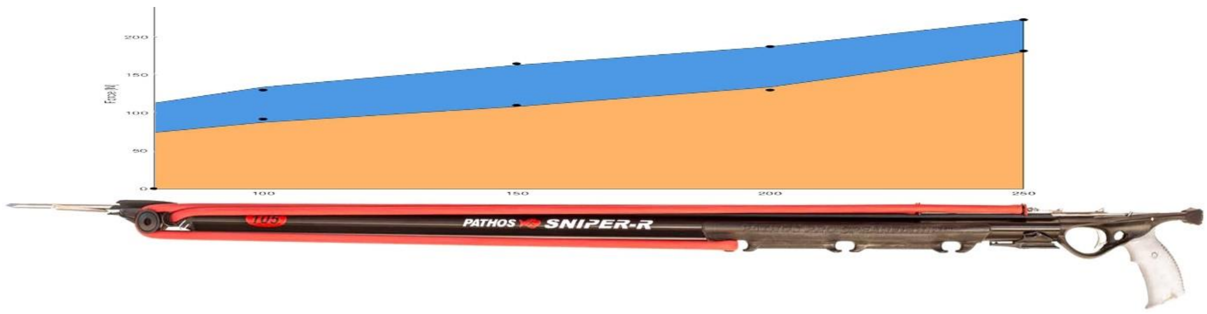


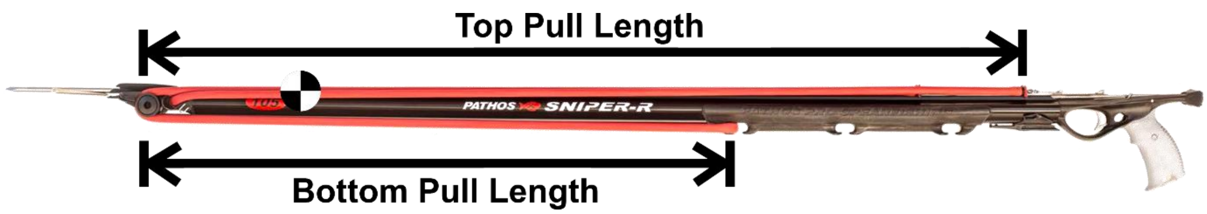
Figure 24 - Roller Speargun Power Diagram

The rubber is still anchored to the shaft end and speargun, so like conventional spearguns it contracts at a rate of half the shaft speed:

$$V_{bands} = 0.5\ V_{spear}$$

Equating the momentum results in issues as, unlike in conventional bands, the centre of mass for roller bands changes direction during firing.

The centre of gravity location before firing can be given by:



$$CoG\ Location\ before\ firing = \frac{Pull_{bottom} + Pull_{top}}{2}$$

Since the rollers, muzzle and wishbone prevent the rubbers from contracting further: the centre of gravity location after firing can be given by:

$$CoG\ Location\ after\ firing = \frac{Pull_{bottom}}{2}$$

The total travel of the roller band CoG can be given by:

$$CoG_{total\_travel} = \frac{Pull_{bottom}}{2} + \frac{Pull_{bottom} + Pull_{top}}{2} - Pull_{bottom}$$

$$CoG_{total\_travel} = Pull_{bottom} + \frac{Pull_{top}}{2} - Pull_{bottom}$$

$$CoG_{total\_travel} = \frac{Pull_{top}}{2}$$

The roller band CoG will always travel in the same net direction as the stock, countering recoil forces if:

$$Pull_{top} > Pull_{bottom}$$

Which is an inherent trait of roller gun design. The net length the CoG travels backwards can be given by subtracting the bottom travel distance by the top travel distance:

$$CoG_{back\_travel} = \frac{Pull_{bottom}}{2} - \left[\frac{Pull_{bottom} + Pull_{top}}{2} - Pull_{bottom}\right]$$

$$CoG_{back\_travel} = Pull_{bottom} - \frac{Pull_{top}}{2}$$

However, this is useless for equating linear equivalent momentum. For this, the proportion of the net rearwards travel must be related to the gross CoG travel as a coefficient called the '*back-fraction'*:

$$\text{Back\_Fraction} = \frac{CoM_{back\_travel}}{CoM_{total\_travel}}$$

$$\text{Back\_Fraction} = \frac{\left[Pull_{bottom} - \frac{Pull_{top}}{2}\right]}{\frac{Pull_{top}}{2}}$$

$$\text{Back\_Fraction} = \frac{(2Pull_{bottom} - Pull_{top})}{Pull_{top}}$$

The back fraction is always negative as rollerguns have a greater top than bottom pull length. The back-fraction coefficient is determined only by the top and bottom pull length dimensions, does not change with different speargun rigging and represents the proportion of the roller band's momentum that travels in the same direction as the stock, countering the spear's momentum.

The back-fraction coefficient can now be used to demonstrate momentum conservation in the horizontal plane:

$$\Delta\, P_{system} = 0$$

$$M_{gun}V_{gun} = (M_{spear}\, V_{spear}) - \text{Back\_Fraction} \times (M_{bands}V_{bands})$$

$$M_{gun}V_{gun} = (M_{spear}\, V_{spear}) - \left[\frac{(2Pull_{bottom} - Pull_{top})}{Pull_{top}}\right](M_{bands} \times V_{bands})\,)$$

Like the conventional speargun, Substituting equation X into the momentum balance below:

$$V_{bands} = 0.5\, V_{spear}$$

$$M_{gun}V_{gun} = (M_{spear}\, V_{spear}) - \left[\frac{(2Pull_{bottom} - Pull_{top})}{Pull_{top}}\right](M_{bands} \times\ 0.5\, V_{spear})$$

$$M_{gun}V_{gun} = V_{spear}(M_{spear} - \left[\frac{(2Pull_{bottom} - Pull_{top})}{Pull_{top}}\right](0.5M_{bands})$$

$$V_{spear} = \frac{M_{gun}V_{gun}}{\left[M_{spear} - \left[\frac{(2Pull_{bottom} - Pull_{top})}{Pull_{top}}\right](\frac{M_{bands}}{2})\right]}$$

The kinetic energy in the system is also conserved in the horizontal plane:

$$\Delta\, E_{system} = 0$$

$$E_{system} = \frac{M_{gun}{V_{gun}}^2}{2} + \frac{M_{bands}{V_{bands}}^2}{2} + \frac{M_{spear}{V_{spear}}^2}{2}$$

Substituting equation X into the energy balance below:

$$2E_{system} = M_{gun}{V_{gun}}^2 + M_{bands}(0.5\, V_{spear})^2 + M_{spear}{V_{spear}}^2$$

$$M_{gun}{V_{gun}}^2 + 0.25\, M_{bands}{V_{spear}}^2 + M_{spear}{V_{spear}}^2 - 2E_{system} = 0$$

$$M_{gun}{V_{gun}}^2 + {V_{spear}}^2\,(\,0.25\, M_{bands} + M_{spear}) - 2E_{system} = 0$$

Substituting equation X's expression for spear velocity results an expression for recoil velocity in terms of component masses and system energy:

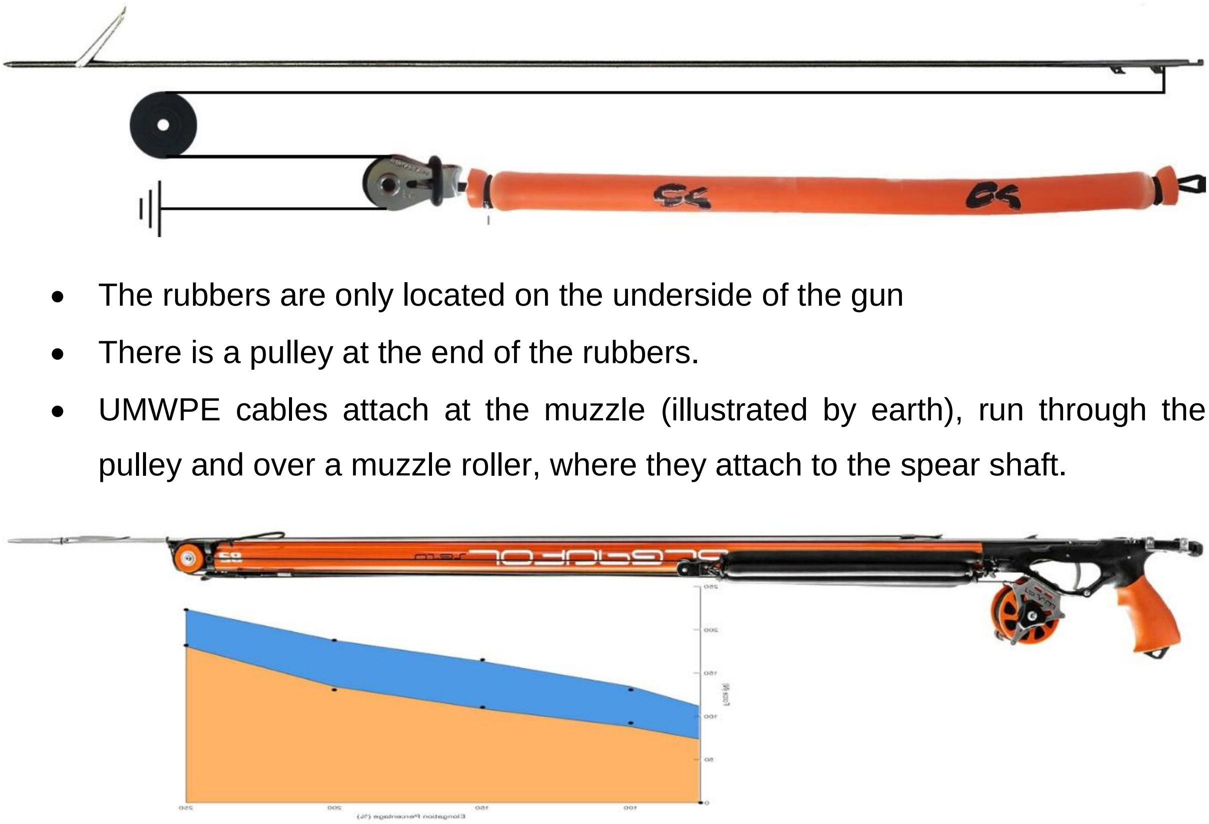
$$M_{gun}V_{gun}^{2} + \left[\frac{M_{gun}\,V_{gun}}{\left(M_{spear} - (Back\_Fraction)\frac{M_{bands}}{2}\right)}\right]^{2}\left[\frac{M_{bands}}{4} + M_{spear}\right] - 2E_{system} = 0$$

This equation can be solved for gun velocity. The result is fed into equation X to find spear velocity and the result for spear velocity halved to find band velocity.

All results can be checked by adding component kinetic energies and comparing the result to system energy.

4.3 Inverted Spearguns

Inverted speerguns are different to both conventional and roller speerguns:



- The rubbers are only located on the underside of the gun
- There is a pulley at the end of the rubbers.
- UMWPE cables attach at the muzzle (illustrated by earth), run through the pulley and over a muzzle roller, where they attach to the spear shaft.

Figure 25 - Inverted Speargun Power Diagram

This enables pretension (like a rollergun) but has a shorter travel distance (like a conventional speargun).The pulley halves the rubber travel and speed but doubles the tension force; this causes the rubber to contract at a rate four times slower than the spear velocity:

$$V_{bands} = 0.25\, V_{spear}$$

Since the force is doubled, a rollergun can have two bands in parallel whilst the user only experiences the pull force of one: this doubles the energy stored. There is no need to incorporate a back fraction as the rubber CoG only travels rearwards.

$$\Delta\, P_{system} = 0$$

$$M_{gun}V_{gun} = (M_{spear}\, V_{spear}) - (M_{bands}V_{bands})$$

Substituting equation X into the momentum balance below:

$$M_{gun}V_{gun} = (M_{spear}V_{spear}) - \left(M_{bands} \times 0.25\, V_{spear}\right)$$

$$M_{gun}V_{gun} = V_{spear}\,(M_{spear} - 0.25\, M_{bands})$$

$$M_{gun}\, V_{gun} = V_{spear}\,(M_{spear} - \tfrac{M_{bands}}{4})$$

$$V_{spear} = \frac{M_{gun}\, V_{gun}}{\left[M_{spear} - \frac{M_{bands}}{4}\right]}$$

The kinetic energy in the system is also conserved in the horizontal plane:

$$\Delta\, E_{system} = 0$$

$$E_{system} = \frac{M_{gun}{V_{gun}}^2}{2} + \frac{M_{bands}{V_{bands}}^2}{2} + \frac{M_{spear}{V_{spear}}^2}{2}$$

Substituting equation X into the energy balance below:

$$2E_{system} = M_{gun}{V_{gun}}^2 + M_{bands}(0.25\, V_{spear})^2 + M_{spear}{V_{spear}}^2$$

$$M_{gun}{V_{gun}}^2 + 0.0625\, M_{bands}{V_{spear}}^2 + M_{spear}{V_{spear}}^2 - 2E_{system} = 0$$

$$M_{gun}{V_{gun}}^2 + {V_{spear}}^2\,(0.0625\, M_{bands} + M_{spear}) - 2E_{system} = 0$$

Substituting equation X's expression for spear velocity results an expression for recoil velocity in terms of component masses and system energy:

$$M_{gun}{V_{gun}}^2 + \left[\frac{M_{gun}\,V_{gun}}{\left(M_{spear} - \frac{M_{bands}}{4}\right)}\right]^2 \left[\frac{M_{bands}}{16} + M_{spear}\right] - 2E_{system} = 0$$

This equation can be solved for gun velocity. The result is fed into equation X to find spear velocity and the result for spear velocity quartered to find band velocity.

All results can be checked by adding component kinetic energies and comparing the result to system ene

4.4 Combining the Equations

It is possible for some or all the power systems listed prior to be combined into the one speargun:

$$V_{spear} = 2\,V_{conventional} = 2\,V_{roller} = 4\,V_{inverted}$$

$$\text{(BF) Back\_Fraction} = \frac{(2Pull_{bottom} - Pull_{top})}{Pull_{top}}$$

Equating the horizontal momentum can give an expression for spear velocity in terms of gun velocity:

$$\Delta\, P_{system} = \sum M\, V = 0$$

$$(M_{spear}\, V_{spear}) - M_{gun}V_{gun} + M_{conventional}\, V_{conventional} - \mathrm{BF}(M_{roller}V_{roller}) - M_{inverted}V_{inverted} = 0$$

Substituting equation X into the momentum balance below:

$$(M_{spear}\, V_{spear}) - M_{gun}V_{gun} + \frac{M_{conventional}\, V_{spear}}{2} - \frac{\mathrm{BF}(M_{roller}V_{spear})}{2} - \frac{M_{inverted}V_{spear}}{4} = 0$$

$$V_{spear}\left[M_{spear} + \frac{M_{conventional}}{2} - \frac{\mathrm{BF}(M_{roller})}{2} - \frac{M_{inverted}}{4}\right] - M_{gun}V_{gun} = 0$$

$$V_{spear} = \frac{M_{gun}V_{gun}}{\left[M_{spear} + \frac{M_{conventional}}{2} - \frac{\mathrm{BF}(M_{roller})}{2} - \frac{M_{inverted}}{4}\right]}$$

The kinetic energy in the system is also conserved in the horizontal plane:

$$\Delta\, E_{system} = 0$$

$$E_{system} = \frac{M_{gun}V_{gun}^{\ 2}}{2} + \frac{M_{spear}V_{spear}^{\ 2}}{2} + \frac{M_{conventional}V_{conventional}^{\ 2}}{2} + \frac{M_{roller}V_{roller}^{\ 2}}{2} + \frac{M_{inverted}V_{inverted}^{\ 2}}{2}$$

$$2E_{system} = M_{gun}V_{gun}^{\ 2} + M_{spear}V_{spear}^{\ 2} + + M_{conventional}V_{conventional}^{\ 2} + M_{roller}V_{roller}^{\ 2} + M_{inverted}V_{inverted}^{\ 2}$$

Substituting equation X into the energy balance below:

$$2E_{system} = M_{gun}V_{gun}^{\ 2} + M_{spear}V_{spear}^{\ 2} + \frac{M_{conventional}V_{spear}^{\ 2}}{4} + \frac{M_{roller}V_{spear}^{\ 2}}{4} + \frac{M_{inverted}V_{spear}^{\ 2}}{16}$$

$$M_{gun}V_{gun}^{\ 2} + V_{spear}^{\ 2}\left[M_{spear} + \frac{M_{conventional}}{4} + \frac{M_{roller}}{4} + \frac{M_{inverted}}{16}\right] - 2E_{system} = 0$$

Substituting equation X's expression for spear velocity results an expression for recoil velocity in terms of component masses and system energy:

$$M_{gun}V_{gun}^{\ 2} + \left[\frac{M_{gun}V_{gun}}{\left[M_{spear} + \frac{M_{conventional}}{2} - \frac{\mathrm{BF}(M_{roller})}{2} - \frac{M_{inverted}}{4}\right]}\right]^2 \left[M_{spear} + \frac{M_{conventional} \ + M_{roller}}{4} + \frac{M_{inverted}}{16}\right] - 2E_{system} = 0$$

This equation can be solved for gun velocity. The result is fed into equation X to find spear velocity and the result for spear velocity fed into equation X to determine component velocities.

Equations X, Y and Z will form the foundation of the simulator cod